THE UNIVERSITY OF TEXAS AT AUSTIN
Dept. of Electrical and Computer Engineering

Final Exam

Date: December 11, 2003

Course: EE 313 Evans/Arifler

Name: SOLUTIONS
Last, First

- The exam is scheduled to last 3 hours.
- Open books and open notes. You may refer to your homework and solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- Turn off all cellular phones and pagers.
- All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.**

Problem	Point Value	Your Score	Topic
1	10		Fourier Series
2	10		Transfer Function
3	10		Laplace Transforms
4	10		z-Transforms
5	15		Signal Energy
6	15		Fourier Transform
7	15		Convolution
8	15		Miscellaneous
Total	100		

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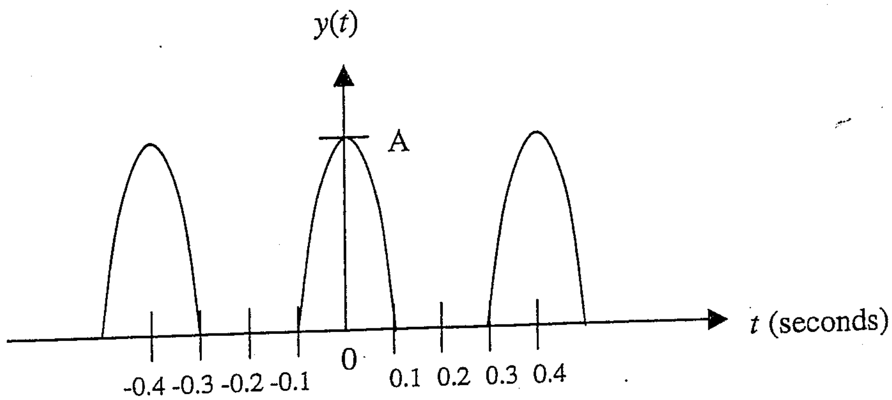
Problem 1 Fourier Series. 10 points.

Consider the *half-sinusoidal* periodic waveform $y(t)$ in the figure below.



(a) (3 points) Write down the period T_0 in seconds, the frequency f_0 in Hertz, and frequency ω_0 in radians/s of $y(t)$.

$T_0 =$ 0.4 s

$f_0 =$ 2.5 Hz

$\omega_0 =$ 5π rad/s

(b) (3 points) Write down the functional expression of $y(t)$ over *one period*.

$$y(t) = \begin{cases} A \cos 5\pi t, & -0.1 < t \le 0.1 \\ 0, & 0.1 \le t < 0.3 \end{cases}$$

(c) (4 points) Find the zero-frequency component (DC value) of $y(t)$.

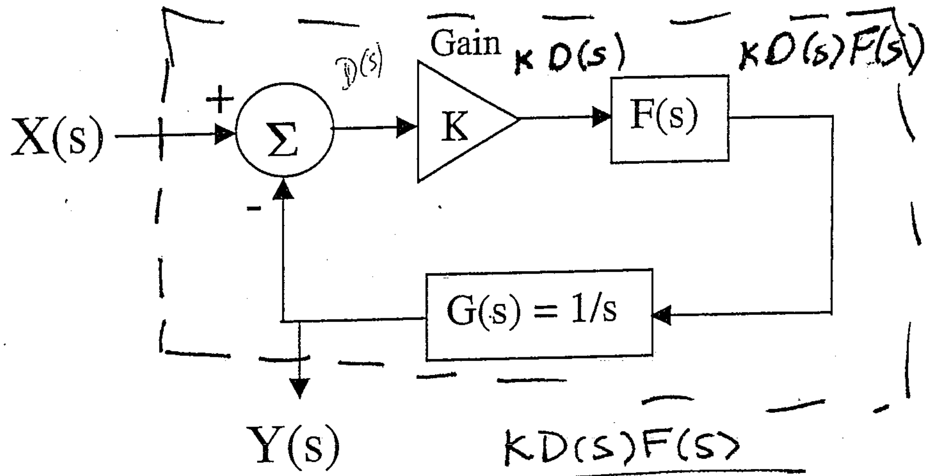
$$a_0 = \frac{1}{T} \int_{-0.1}^{0.1} A \cos 5\pi t \, dt$$

$$= \frac{1}{0.4} \left. \frac{A \sin 5\pi t}{5\pi} \right|_{-0.1}^{0.1} = \frac{A}{0.4 \times 5\pi} \left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2} \right)$$

$$= \frac{A}{2\pi} \times 2 = \frac{A}{\pi}$$

Problem 2 Transfer Function. 10 points.

The block diagram of a linear phase-lock loop is given in the figure below. Find the transfer function $H(s) = \dfrac{Y(s)}{X(s)}$ in terms of K and $F(s)$.



$$\frac{K D(s) F(s)}{s}$$

$$D(s) = X(s) - Y(s)$$

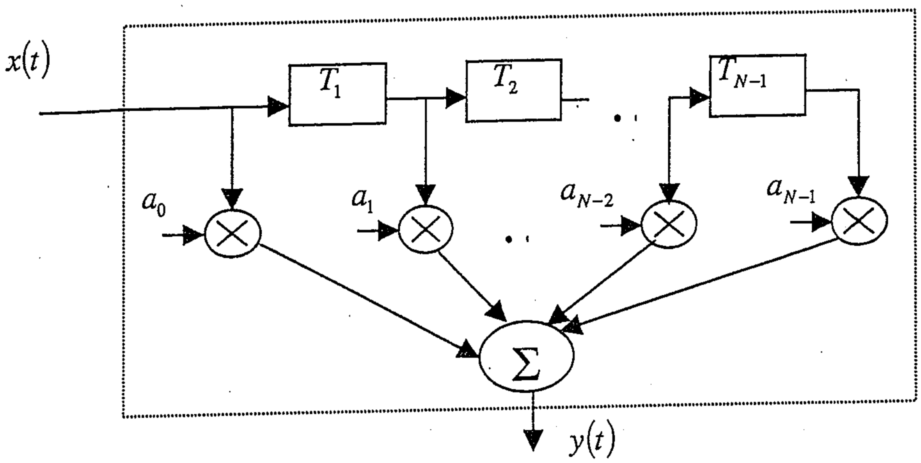
$$Y(s) = \frac{K\,(X(s) - Y(s))\,F(s)}{s}$$

$$Y(s) = \frac{K X(s) F(s)}{s} - \frac{K\,(Y(s) F(s))}{s}$$

$$Y(s)\left[1 + \frac{KF(s)}{s}\right] = \frac{K X(s) F(s)}{s}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K F(s)}{s + K F(s)}$$

Problem 3 Laplace Transforms. 10 points.



For the continuous-time tapped delay line given in the figure above, in which each of the $N-1$ delay blocks has a possibly different delay value, the impulse response is given by

$$h(t) = \sum_{n=0}^{N-1} a_n \delta(t - \tau_n)$$

where $\tau_n = \sum_{m=0}^{n} T_m$ with $T_0 = 0$. It is also given that the system is in zero initial state.

(a) (5 points) What is the Laplace transform of $h(t)$?

$$H(s) = \sum_{n=0}^{N-1} a_n e^{-s\tau_n}$$

(b) (5 points) What is the Laplace transform of the output $Y(s)$ when the input $x(t)$ is the unit step $u(t)$?

$$X(s) = \frac{1}{s}, \quad Y(s) = H(s)\,X(s)$$

$$Y(s) = \sum_{n=0}^{N-1} \frac{a_n e^{-s\tau_n}}{s}$$

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Problem 4 z-Transforms. 10 points.

Consider the following z-transform pair:

$$x[k] = \gamma^k u[k] \leftrightarrow X[z] = \frac{z}{z-\gamma}, \qquad |z| > |\gamma|$$

(a) (2 points) Find the derivative of $x[k]$ with respect to γ.

$$\frac{\partial x[k]}{\partial \gamma} = k\gamma^{k-1}$$

(b) (2 points) Find the derivative of $X[z]$ with respect to γ.

$$\frac{\partial X[z]}{\partial \gamma} = \frac{z}{(z-\gamma)^2}$$

(c) (3 points) *Using only parts (a) and (b)*, and the properties of the z-transform, what is the z-transform of $y[k] = k\gamma^k u[k]$?

Multiply each side by γ.

$$\gamma k \gamma^{k-1} u[k] \longleftrightarrow \frac{\gamma z}{(z-\gamma)^2}$$

$$k\gamma^k u[k] \longleftrightarrow \frac{\gamma z}{(z-\gamma)^2}$$

(d) (3 points) Given that the energy of a sequence $y[k]$ is $\sum_{k=-\infty}^{\infty} y[k]y^*[k]$. Can $y[k] = k\gamma^k u[k]$ have finite energy? If it can, for what values of γ? Justify your answer.

If $\gamma < 1$, then γ^{2k} decays faster than k^2, and $y[k]$ has finite energy.

(Note: $\sum$ converges for $\gamma < 1$)

M-17

Problem 5 Signal Energy. 15 points.

Recall that the energy of a real signal $f(t)$ is defined as the energy dissipated when a voltage $f(t)$ is applied across (or if a current $f(t)$ is passed through) a 1Ω resistor. Find the energy of the signal given by $f(t) = 20\,e^{-10t}u(t)$ in the interval:

(a) (7 points) $-0.1 < t < 0.1$ s.

$$\int_0^{0.1} 20^2 e^{-20t}\, dt = \frac{20^2}{-20} e^{-20t}\Big|_0^{0.1}$$

$$= -20\left(e^{-2} - 1\right) \cong 17.29$$

(b) (8 points) $-10 < \omega < 10$ radians/s.
Hint: You may find the following identity useful for this part of the problem:

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

(3) $$\mathcal{F}\{20e^{-10t}u(t)\} = \frac{20}{10+j\omega}$$

(3) $$\frac{1}{2\pi}\int_{-10}^{10} \frac{20^2}{(10+j\omega)(10-j\omega)}\, d\omega = \frac{20^2}{2\pi} \times 2 \int_0^{10} \frac{1}{100+\omega^2}\, d\omega$$

$$= \frac{1}{10}\,\frac{20^2}{2\pi} \times 2 \;\tan^{-1}\frac{\omega}{10}\Big|_0^{10}$$

$$= \frac{20^2}{10\pi} \tan^{-1} 1$$

(2) $$= \frac{1}{10}\,\frac{20^2}{\cancel{\pi}}\,\frac{\cancel{\pi}}{4} = 10$$

M-18

Problem 6 Fourier Transforms. 15 points.

The frequency response of a linear time-invariant system whose input is $f(t)$ and output is $y(t)$ is given by

$$H(\omega) = \frac{10 - j2\omega}{4 + j\omega}.$$

Find the *time-domain* output $y(t)$ if the input $f(t)$ is given by:

(a) (5 points) The unit impulse $\delta(t)$.

$$Y(\omega) = \frac{10}{4+j\omega} - \frac{j2\omega}{4+j\omega} = \frac{10}{4+j\omega} - \left(2 - \frac{8}{4+j\omega}\right)$$

$$= \frac{18}{4+j\omega} - 2$$

(5) $$y(t) = 18e^{-4t}u(t) - 2\delta(t)$$

(b) (5 points) The unit step $u(t)$.

(2) $$Y(\omega) = \frac{10-j2\omega}{4+j\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \frac{10-j2\omega}{4+j\omega}\pi\delta(\omega) + \frac{10-j2\omega}{4+j\omega}\frac{1}{j\omega}$$

$$= \frac{10\pi}{4}\delta(\omega) + \frac{10-j2\omega}{(4+j\omega)j\omega}$$

$$= 2.5\pi\delta(\omega) + \frac{10}{(4+j\omega)j\omega} - \frac{2\cancel{j\omega}}{(4+j\omega)\cancel{j\omega}}$$

$$= 2.5\pi\delta(\omega) - \frac{2.5}{4+j\omega} + \frac{2.5}{j\omega} - \frac{2}{4+j\omega}$$

$$= 2.5\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - \frac{4.5}{4+j\omega}$$

$$y(t) = 2.5u(t) - 4.5e^{-4t}u(t)$$

(c) (5 points) A sinusoid $\cos(t)$.

(3)

$$|H(\omega)| = \left|\frac{10-j2}{4+j}\right| = |2.235 - j1.059| = 2.473$$

$$\angle H(\omega) = -0.442 \text{ rad/s} \quad \text{or} \quad -25.35^\circ$$

$$y(t) = 2.473\cos(t - 0.442)$$

$$= 2.24\cos t + 1.06\sin t$$

M-19

Problem 7 Convolution. 15 points.

For two *real* signals $f(t)$ and $h(t)$, the correlation $r(t)$ is given by

$$r(t) = \int_{-\infty}^{\infty} f(\tau)h(t+\tau)d\tau.$$

(a) (4 points) Express $r(t)$ as a convolution operation between $f(t)$ and $h(t)$.

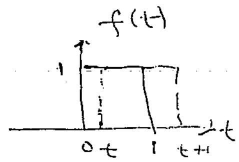
$$r(t) = f(t) * h(-t)$$

(b) (4 points) What property should either $f(t)$ or $h(t)$ have in order for $r(t) = f(t) * h(t)$?

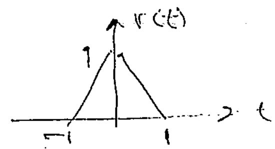
Either $f(t)$ or $h(t)$ is even.

(c) (4 points) Find $r(t)$ when $f(t) = h(t) = \text{rect}(t - 1/2)$. Recall

$$\text{rect}(t-1/2) = \begin{cases} 1, & 0 < t < 1, \\ 0, & \text{otherwise.} \end{cases}$$



$$r(t) = \begin{cases} 0, & t < -1 \\ \int_0^{t+1} 1\, d\tau = t+1, & -1 < t < 0 \\ \int_t^1 1\, d\tau = 1-t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$



(d) (3 points) When $f(t) = h(t)$, what is the relationship between $r(t)$ and the energy of signal $f(t)$?

Energy of $f(t) = r(0)$

M-20

Problem 8 Miscellaneous short answer questions. 15 points.

(a) (3 points) List two reasons why modulation is necessary in communication systems.

(1) Radio stations, simultaneous broadcast.

(2) Practical antennas can be designed -
(Effective radiation of power → Antenna size ~ wavelength of signal)

(b) (4 points) Is it possible for a signal to have an arbitrarily large (and possibly infinite) duration in the time domain and an arbitrarily large (and possibly infinite) bandwidth in the frequency domain at the same time? Justify your answer.

No. By reciprocity of signal duration and its bandwidth.

Yes. Non time limited signals can have non limited bandwidth. Both okay, if justified.

(c) (4 points) Is it possible to have a stable linear system which is non-causal? Justify your answer.

Yes. Stability and causality are independent of each other.

(d) (4 points) Suppose you are given the energy spectral density $|F(\omega)|^2$ of a signal $f(t)$. Is it possible to recover $f(t)$ from the knowledge of $|F(\omega)|^2$? Justify your answer.

No. No phase information in $|F(\omega)|^2$.

M-21

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